Qt-Eigenvalue Theory of Third-Order Quaternion Tensors

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Background

A tensor is a higher-order extension of a matrix and can be regarded as a multidimensional array. After Kilmer and Martin [1] introduced T-product, more outstanding research outputs related to T-product have been well developed.

- Miao et al. [2] brought up the definition of T-eigenvalue, listed some useful decompositions and polynomial functions related to T-product of tensors.
- Liu and Jin [3] studied T-eigenvalues of third-order tensors and showed some T-eigenvalue inequalities for Hermitian tensors.
- Hao et al. [4] proposed a principal component analysis approach for real third-order tensors based on the TSVD proposed by Kilmer et al., and applied it to facial recognition.

1 M. Kilmer, C. Martin, Factorization strategies for third-order tensors, *Linear Algebra Appl.* 435 (2011) 641–658.

2 Y. Miao, L. Qi, Y. Wei, T-Jordan canonical form and T-Drazin inverse based on the T-product, *Commun. Appl. Math. Comput.* 3 (2021) 201-220.

3 W.H. Liu, X.Q. Jin, A study on T-eigenvalues of third-order tensors, *Linear Algebra Appl.* 612 (2021) 357-374.

4 N. Hao, M. E. Kilmer, K. Braman, R. C. Hoover, Facial recognition using tensor-tensor decomposition, *SIAM J. Imag. Sci.* 6 (2013)437–463.

If the elements of a tensor $\mathcal{A}$ are quaternions, then $\mathcal{A}$ is called a quaternion tensor. Quaternion tensor has been widely investigated in computer graphics, robotics, signal processing, optimization problems, artificial intelligence and other fields.

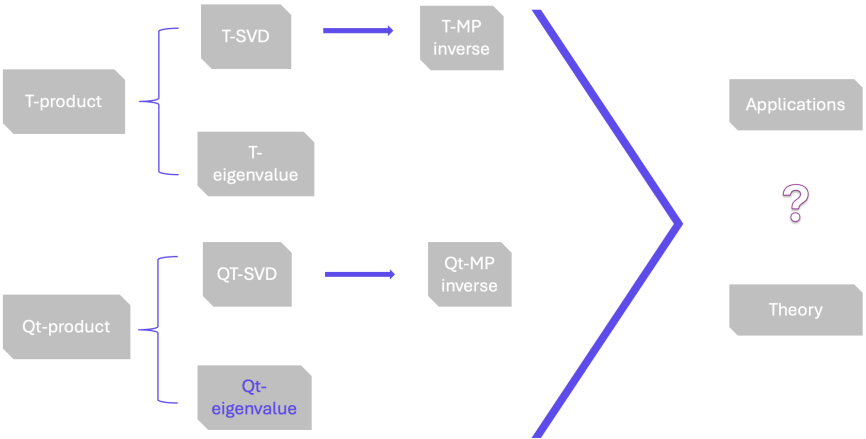
- He et al. [1] investigated the definitions, propositions and calculation of eigenvalues of quaternion tensor based on Einstein product, also showed some practical applications.
- Qin et al. [2] investigated singular decomposition of 3rd-order quaternion tensor based on Qt-product.
- Zuo and Ma [3] investigated the Moore-Penrose inverse of quaternion tensors under the Qt-product.

1 Z.H. He, X.X. Wang, Y.F. Zhao, Eigenvalues of quaternion tensors with applications to color video processing, *J. Sci. Comput* 94 (2023) 1.

2 Z.Z. Qin, Z.Y. Ming, L.P. Zhang, Singular value decomposition of third order quaternion tensors, *Appl. Math. Lett.* 123 (2022) 107597.

3 S. Zuo, H. Ma, The perturbation of the Moore–Penrose inverse of quaternion tensor via the QT-product, *J. Appl. Math. Comput. 69(5)* (2023) 3937-3967.

Motivation


T-product
T-SVD
T-eigenvalue
T-MP inverse
Qt-product
QT-SVD
Qt-eigenvalue
Qt-MP inverse
Applications
?
Theory

Notation

The capitals, such as A, are used to denote matrices. The calligraphic letters, such as $\mathcal{A}$, are used to denote tensors. Let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ represent real field, complex field and quaternions, respectively.

The quaternion algebra, $\mathbb{H} = \{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} | \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$.
If $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a quaternion, then $a^* = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$ and $a^{-1} = a^*/|a|$, where $|a| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$.

For any third-order quaternion tensor $\mathcal{A} \in \mathbb{H}^{n_1 \times n_2 \times n_3}$,

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1\mathbf{i} + \mathcal{A}_2\mathbf{j} + \mathcal{A}_3\mathbf{k} = \mathcal{A}_{1i} + \mathbf{j}\mathcal{A}_{jk},$$

where $\mathcal{A}_{1i} = \mathcal{A}_0 + \mathcal{A}_1\mathbf{i}$, $\mathcal{A}_{jk} = \mathcal{A}_2 - \mathcal{A}_3\mathbf{i}$ and $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3 \in \mathbb{R}^{n_1 \times n_2 \times n_3}$.

The operations *circ*, *uncirc*, *unfold* and *fold* are defined as follows:

$$circ(\mathcal{A}) := \begin{bmatrix} A^{(1)} & A^{(n_3)} & \dots & A^{(2)} \\ A^{(2)} & A^{(1)} & \dots & A^{(3)} \\ & & \ddots & \\ A^{(n_3)} & A^{(n_3-1)} & \dots & A^{(1)} \end{bmatrix}, \quad unfold(\mathcal{A}) := \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(n_3)} \end{bmatrix},$$

$$fold(unfold(\mathcal{A})) := \mathcal{A} \text{ and } uncirc(circ(\mathcal{A})) := \mathcal{A}.$$

It is well known that any complex circulant matrix can be diagonalized by the normalized Discrete Fourier Transformation (DFT). In other words,

$$(F_{n_3} \otimes I_{n_1})(circ(\mathcal{A}))(F_{n_3}^* \otimes I_{n_2}) = \begin{bmatrix} \widehat{A^{(1)}} & & & \\ & \widehat{A^{(2)}} & & \\ & & \ddots & \\ & & & \widehat{A^{(n_3)}} \end{bmatrix} = diag(\widehat{\mathcal{A}}),$$

for any $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$.

Qt-product

If $\mathcal{A} \in \mathbb{H}^{n_1 \times n_2 \times m}$ and $\mathcal{B} \in \mathbb{H}^{n_2 \times n_3 \times m}$, then the Qt-product of $\mathcal{A}$ and $\mathcal{B}$ can be defined as

$$\mathcal{A} \star \mathcal{B} = fold\{[circ(\mathcal{A}_{1i}) + \mathbf{j}circ(\mathcal{A}_{jk})(P_m \otimes I_{n_2})][unfold(\mathcal{B})]\},$$

where $unfold(\mathcal{B}) = unfold(\mathcal{B}_{1i}) + \mathbf{j}unfold(B_{jk})$.

After taking normalized Discrete Fourier Transformation on $\mathcal{A}$, we have

$$unfold(\widehat{\mathcal{A}}) = \sqrt{m}(F_m \times I_{n_1})unfold(\mathcal{A}),$$

and

$$diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{A}_{1i}}) + \mathbf{j}(P_m \otimes I_{n_1})diag(\widehat{\mathcal{A}_{jk}})(P_m \otimes I_{n_2}).$$

Z.Z. Qin, Z.Y. Ming, L.P. Zhang, Singular value decomposition of third order quaternion tensors, *Appl. Math. Lett.* 123 (2022) 107597.

The conjugate transpose of a quaternion tensor $\mathcal{A} = \mathcal{A}_{1i} + \mathbf{j}\mathcal{A}_{jk} \in \mathbb{H}^{n_1 \times n_2 \times n_3}$ is denoted as $\mathcal{A}^* \in \mathbb{H}^{n_2 \times n_1 \times n_3}$, satisfying

$$unfold(\mathcal{A}^*) = unfold(\mathcal{A}_{1i}^*) - (P_m \otimes I_{n_2}) unfold(\mathcal{A}_{jk}^*)\mathbf{j}.$$

We can derive that

$$(diag(\widehat{\mathcal{A}}))^* = diag(\widehat{\mathcal{A}^*})$$

Lemma (Qin et al. 2022)

Let $\mathcal{A} \in \mathbb{H}^{n_1 \times n_2 \times m}$, $\mathcal{B} \in \mathbb{H}^{n_2 \times n_3 \times m}$, $\mathcal{C} \in \mathbb{H}^{n_1 \times n_3 \times m}$, $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$, $\hat{\mathcal{C}}$ are the DFT of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, respectively. Then

$$diag(\widehat{\mathcal{C}}) = diag(\widehat{\mathcal{A}}) diag(\widehat{\mathcal{B}}) \Leftrightarrow \mathcal{C} = \mathcal{A} \star \mathcal{B}.$$

These yield

$$(\mathcal{A} \star \mathcal{B})^* = \mathcal{B}^* \star \mathcal{A}^*.$$

Definition

Let $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$. Then

1. $\mathcal{A}$ is a Qt-normal tensor if and only if $\mathcal{A}^* \star \mathcal{A} = \mathcal{A} \star \mathcal{A}^*$.
2. $\mathcal{A}$ is Qt-hermitian tensor if and only if $\mathcal{A}^* = \mathcal{A}$.
3. $\mathcal{A}$ is Qt-unitary tensor if and only if $\mathcal{A}^* \star \mathcal{A} = \mathcal{A} \star \mathcal{A}^* = \mathcal{I}$.
4. $\mathcal{A}$ is identity tensor whose first frontal slice is the identity matrix and whose other slices all zero. It is denoted by $\mathcal{I}_{nnm}$.
5. $\mathcal{A}$ is Qt-invertiable if and only if there is a quaternion tensor $\mathcal{B} \in \mathbb{H}^{n\times n\times m}$ such that $\mathcal{A} \star \mathcal{B} = \mathcal{B} \star \mathcal{A} = \mathcal{I}$. We define $\mathcal{A}^{-1} = \mathcal{B}$.

Lemma (Miao et al. 2021)

Suppose $A^{(1)}, A^{(2)}, \cdots, A^{(m)}, B^{(1)}, B^{(2)}, \cdots, B^{(m)} \in \mathbb{C}^{n\times n}$ *are complex matrices satisfying*

$$\begin{bmatrix} A^{(1)} & A^{(m)} & \cdots & A^{(2)} \\ A^{(2)} & A^{(1)} & \cdots & A^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(m)} & A^{(m-1)} & \cdots & A^{(1)} \end{bmatrix} = (F_m \otimes I_n) \begin{pmatrix} B^{(1)} & & & \\ & B^{(2)} & & \\ & & \ddots & \\ & & & B^{(m)} \end{pmatrix} (F_m^* \otimes I_n)$$

where F_m *is the discrete Fourier matrix of size* $m \times m$*. Then* $B^{(1)}, B^{(2)}, \cdots, B^{(m)}$ *are diagonal (sub-diagonal, upper-triangular, lower-triangular, symmetry) matrices if and only if* $A^{(1)}, A^{(2)}, \cdots, A^{(m)}$ *are diagonal (sub-diagonal, upper-triangular, lower-triangular, symmetry) matrices.*

Y. Miao, L. Qi, Y. Wei, T-Jordan canonical form and T-Drazin inverse based on the T-product, *Commun. Appl. Math. Comput.* 3 (2021) 201-220.

Corollary

Suppose that $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$ *is a quaternion tensor such that*

$$circ(\mathcal{A}) = circ(\mathcal{A}_{1i}) + \mathbf{j}circ(\mathcal{A}_{jk})$$

and

$$\begin{aligned} diag(\widehat{\mathcal{A}}) &= diag(\widehat{\mathcal{A}_{1i}}) + \mathbf{j}(P_m \otimes I_{n_1})diag(\widehat{\mathcal{A}_{jk}})(P_m \otimes I_{n_2}) \\ &= diag(\widehat{\mathcal{A}^{(1)}}, \widehat{\mathcal{A}^{(2)}}, \cdots, \widehat{\mathcal{A}^{(m)}}). \end{aligned}$$

Then $\widehat{\mathcal{A}^{(1)}}$, $\widehat{\mathcal{A}^{(2)}}$, $\cdots$, $\widehat{\mathcal{A}^{(m)}}$ *are diagonal (sub-diagonal, upper-triangular, lower-triangular, upper-bidiagonal) matrices if and only if* $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$, $\cdots$, $\mathcal{A}^{(m)}$ *are diagonal (sub-diagonal, upper-triangular, lower-triangular, upper-bidiagonal) matrices.*

Eigenvalues of Third-order Quaternion Tensor Based on Qt-product

Let $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$ is a 3rd-order tensor. If there exists a nonzero tensor $\mathcal{X} \in \mathbb{H}^{n\times 1\times m}$ and a quaternion λ satisfying

$$\mathcal{A} \star \mathcal{X} = \mathcal{X}\lambda,$$

then we say that $(\mathcal{X}, \lambda)$ is a right eigenpair of quaternion tensor $\mathcal{A}$, $\mathcal{X}$ is the eigen-tensor associated with eigenvalue λ, and we call λ is a right Qt-eigenvalue of quaternion tensor $\mathcal{A}$.

If λ is a right Qt-eigenvalue of $\mathcal{A}$, then for any $q \in [\lambda]$, the equivalence class of λ, is a right Qt-eigenvalue of $\mathcal{A}$.

Theorem

If λ is a right Qt-eigenvalue of quaternion tensor $\mathcal{A}$, then

1. λ is a right eigenvalue of quaternion matrix $(circ(\mathcal{A}_{1i})) + \mathbf{j}(circ(\mathcal{A}_{jk}))(P_m \otimes I_n)$ and $unfold(\mathcal{X})$ is a corresponding eigenvector.
2. When λ is a complex number, it is also a right eigenvalue of $diag(\widehat{\mathcal{A}})$ and $(F_m^* \otimes I_n)(unfold(\mathcal{X}))(F_m \otimes I_n)$ is a corresponding eigenvector.

Qt-Jordan Canonical Form and Qt-function of Third-order Quaternion Tensor

Suppose that $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$. Then $\mathcal{A}$ has Qt-inverse if and only if $diag(\widehat{\mathcal{A}})$ is invertible. Furthermore,

$$diag(\widehat{\mathcal{A}^{-1}}) = diag(\widehat{\mathcal{A}})^{-1}.$$

Suppose $\mathcal{A}, \mathcal{B} \in \mathbb{H}^{n\times n\times m}$ are two F-square third-order quaternion tensors. We call that $\mathcal{A}$ is Qt-similar to $\mathcal{B}$ if there exists an Qt-invertible quaternion tensor $\mathcal{P} \in \mathbb{H}^{n\times n\times m}$ satisfying

$$\mathcal{A} = \mathcal{P}^{-1} \star \mathcal{B} \star \mathcal{P}$$

Theorem (Qt-Jordan Canonical form)

Let $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$. Then there exist Qt-invertible quaternion tensor $\mathcal{X} \in \mathbb{H}^{n\times n\times m}$ which can transform $\mathcal{A}$ into a F-upper-bi-diagonal tensor $\mathcal{J} \in \mathbb{H}^{n\times n\times m}$ such that

$$\mathcal{A} = \mathcal{X}^{-1} \star \mathcal{J} \star \mathcal{X}$$

where the frontal slices of $\mathcal{J}^{(i)}$ is a Jordan canonical form with diagonal elements are the standard the Qt-eigenvalues of quaternion tensor $\mathcal{A}$.

Quaternion tensor $\mathcal{J}$ is called the Qt-Jordan canonical form of the quaternion tensor $\mathcal{A}$.

Definition (Qt-function)

Let $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$. Then the quaternion tensor Qt-function of $\mathcal{A}$ is defined by

$$f(\mathcal{A}) := fold(f([circ(\mathcal{A}_{1i}) + \mathbf{j}circ(\mathcal{A}_{jk})(P_m \otimes I_n)])[unfold(\mathcal{I})])$$

It can be proved that

$$f(diag(\widehat{\mathcal{A}})) = diag(\widehat{f(\mathcal{A})}),$$

holds for all polynomial functions f with real coefficients.

If there exists invertible quaternion tensor $\mathcal{X} \in \mathbb{H}^{n\times n\times m}$ satisfying

$$\mathcal{A} = \mathcal{X}^{-1} \star \mathcal{J} \star \mathcal{X},$$

then

$$f(\mathcal{A}) = \mathcal{X}^{-1} \star f(\mathcal{J}) \star \mathcal{X},$$

where $\mathcal{J}$ is the Qt-Jordan canonical form of the quaternion tensor $\mathcal{A}$.

Let $\mathcal{A} \in \mathbb{H}^{n\times n\times s}$ be a third-order quaternion tensor. Then

$$p(\mathcal{A}) = \sum_{k=0}^{m} a_k \mathcal{A}^k = a_m \mathcal{A}^m + a_{m-1}\mathcal{A}^{m-1} + \cdots + a_1 \mathcal{A} + a_0 \mathcal{I}$$

is the polynomial of the third-order quaternion tensor $\mathcal{A}$ with real coefficient and whose degree is m, where $a_m \neq 0$.

Also, we denote that

$$p(\mathcal{A}) = \sum_{k=0}^{\infty} a_k \mathcal{A}^k = a_0 \mathcal{I} + a_1 \mathcal{A} + a_2 \mathcal{A}^2 + \cdots$$

a power series of the third-order quaternion tensor $\mathcal{A}$ with real coefficient.

Theorem (Third-order Quaternion Tensor Series Convergence)

Let $\mathcal{A} \in \mathbb{H}^{n\times n\times s}$ be a third-order quaternion tensor and $f : \mathbb{R} \to \mathbb{R}$ be a power series given by

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

The convergence radius of f is denoted by $\rho \in \mathbb{R}$. Suppose the maximum Qt-eigenvalue by module of third-order quaternion tensor $\mathcal{A}$ is

$$\lambda = \max |\lambda_j^{(i)}|, i = 1, 2, \cdots, s, j = 1, 2, \cdots, n.$$

If $\rho > \lambda$, then the third-order quaternion tensor power series converges and the Qt-eigenvalues of third-order quaternion tensor $\sum_{k=0}^{\infty} a_k \mathcal{A}^k$ are

$$\sum_{k=0}^{\infty} a_k (\lambda_j^{(i)})^k, \quad i = 1, 2, \cdots, s, \quad j = 1, 2, \cdots, n.$$

Examples

$$\exp(\mathcal{A}) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathcal{A}^k$$

$$\sin(\mathcal{A}) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!}\mathcal{A}^{2k-1}$$

$$\cos(\mathcal{A}) = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!}\mathcal{A}^{2k}$$

Qt-hermitian and Qt-positive (Semi)definite Tensor

Recall that $\mathcal{A}$ is Qt-hermitian tensor if and only if $\mathcal{A}^* = \mathcal{A}$.

Theorem

Let $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$. $\mathcal{A}$ is Qt-hermitian if and only if diag$(\widehat{\mathcal{A}})$ is hermitian.

Let $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$ is Qt-hermitian. We say that $\mathcal{A}$ is a Qt-positive (semi)definite tensor if and only if all Qt-eigenvalues of $\mathcal{A}$ are nonnegative (positive).

Theorem

A third-order quaternion tensor is Qt-positive semi(definite) if and only if the diag$(\widehat{\mathcal{A}})$ is a positive semi(definite) quaternion matrix.

Let $\mathcal{A} \in \mathbb{H}^{m\times n\times t}$ be a third-order quaternion tensor. Then $\mathcal{A}^* \star \mathcal{A}$ and $\mathcal{A} \star \mathcal{A}^*$ are Qt-positive semidefinite.

Qt-Schur Decomposition and Qt-Spectral Theorem

Theorem (Qt-Schur Decomposition)

Let $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$ be an F-square third-order quaternion tensor. Then there are Qt-unitary tensors $\mathcal{U}$ and F-upper-triangular quaternion tensor $\mathcal{T}$ such that

$$\mathcal{A} = \mathcal{U}^* \star \mathcal{T} \star \mathcal{U},$$

where each diagonal entry of each front slice of $\mathcal{T}$ is an standard Qt-eigenvalue of $\mathcal{A}$.

Qt-Spectral Theorem

If $\mathcal{A} \in \mathbb{H}^{n\times n\times m}$ be Qt-normal (Qt-hermitian, Qt-positive semidefinite, Qt-positive definite), then there are Qt-unitary tensors $\mathcal{U}$ and F-diagonal (real, nonnegative, postive) tensor $\mathcal{D}$ such that

$$\mathcal{A} = \mathcal{U}^* \star \mathcal{D} \star \mathcal{U},$$

where each diagonal entry of each front slice of $\mathcal{D}$ is an Qt-eigenvalue of $\mathcal{A}$.

Numerical Experiments on Qt-PCA-SD

A pure imaginary quaternion can present color pixel:

$$q = r\mathbf{i} + g\mathbf{j} + b\mathbf{k}$$

Supposing that we have a color video and every color pixel of the video can be presented by a pure imaginary quaternion. We can get a third-order quaternion tensor $\mathcal{V}$ and a frontal slice represents a frame of the color video.

Experiment 1: Color Video Compression and Reconstruction Using Qt-PCA-SD

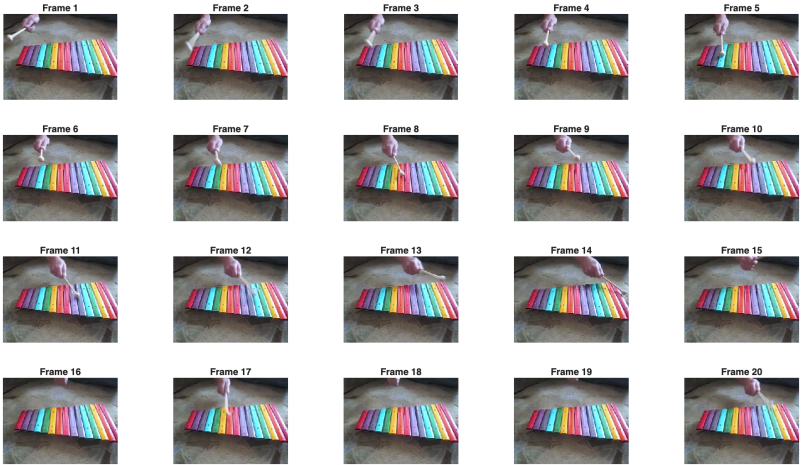


Figure: Original frames extracted from the video xylophone.mp4

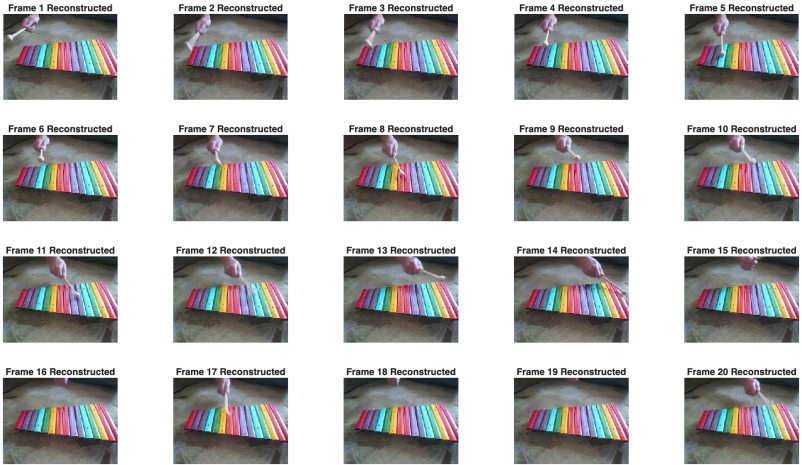


Figure: Reconstructed frames using 100 principal components.

Experiment 2: Color Frames Compression and Reconstruction Using Qt-PCA-SD

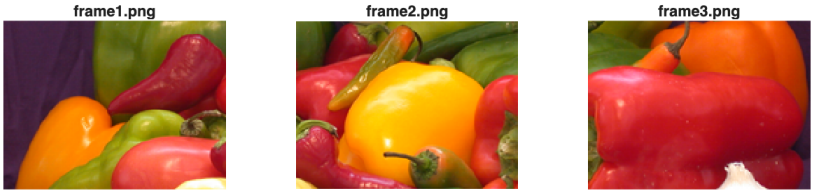


Figure: Original frames.



Figure: Reconstructed frames using 100 principal components.

Experiment 3: Quantitative Comparison between Qt-PCA-SD and Qt-PCA-SVD

Table: Qt-PCA-SD

Frame	PSNR	SSIM	MSE
1	25.09	0.8294	0.003099
2	34.69	0.9905	0.000339
3	32.30	0.9829	0.000589
Avg.	**30.69**	**0.9343**	**0.001342**

Table: Qt-PCA-SVD

Frame	PSNR	SSIM	MSE
1	25.09	0.8286	0.003097
2	34.73	0.9906	0.000337
3	32.26	0.9827	0.000594
Avg.	**30.69**	**0.9340**	**0.001343**

Why Qt-PCA-SD is better than Qt-PCA-SVD?

- Explicit measure of variance contribution for each principal component.
- Interpretable and quantitative selection of the most informative components.
- Better control over dimensionality reduction and more insight into the intrinsic structure of the data.

Thank you for your attention.

Questions and discussions are welcome!