Generalized Eigenvalues of Third-Order Quaternion Tensors and Multilinear Discriminant Analysis Under Qt-Product

Jie Tian

Department of Mathematics and Statistics
University of Nevada, Reno

2026 SIAM Annual Meeting (AN26)
July 6–10, 2026 · Cleveland, Ohio

Motivation

Limitations of classical LDA for color images and videos classfication:

- Flattens tensors to vectors $\Rightarrow$ loses spatial/channel structure
- Sample size $\ll$ feature dimension: within-class scatter S_W is singular (*small-sample-size problem*)
- Requires PCA pre-processing as a workaround
- Multimodal channels treated *independently*

Our Approach

Encode correlated channels (color images, color videos) into a single **quaternion tensor** and perform discriminant analysis directly via the **Qt-product**.

Key Advantages

- No Qt-PCA or PCA for pre-processing
- Cross-channel correlation is preserved
- Tensor-level regularization is provided
- Unified multimodal representation is provided

Quaternions and Qt-Product

Quaternion Ring $\mathbb{H}$

$q = a_0 + a_1 i + a_2 j + a_3 k, \quad a_\ell \in \mathbb{R},\ i^2 = j^2 = k^2 = ijk = -1$

For $\mathcal{A} \in \mathbb{H}^{n_1 \times n_2 \times n_3}$, write

$$\mathcal{A} = \underbrace{\mathcal{A}_0 + \mathcal{A}_1 i}_{\mathcal{A}_{1i}} + j \underbrace{(\mathcal{A}_2 - \mathcal{A}_3 i)}_{\mathcal{A}_{jk}}$$

Qt-Product (Qin et al., 2022)

For $\mathcal{A} \in \mathbb{H}^{n_1 \times n_2 \times m}$, $\mathcal{B} \in \mathbb{H}^{n_2 \times n_3 \times m}$:

$$\mathcal{A} \star \mathcal{B} = \operatorname{fold}\big\{[\operatorname{circ}(\mathcal{A}_{1i}) + j \operatorname{circ}(\mathcal{A}_{jk})(P_m \otimes I)] \operatorname{unfold}(\mathcal{B})\big\}$$

DFT Diagonalization

$$\mathcal{C} = \mathcal{A} \star \mathcal{B} \iff \widehat{C}^{(k)} = \widehat{A}^{(k)} \widehat{B}^{(k)}$$

Generalized Qt-Eigenvalue Problem

Let $\mathcal{A}, \mathcal{B} \in \mathbb{H}^{n\times n\times m}$. The generalized right eigenvalues of the pair $(\mathcal{A}, \mathcal{B})$ are elements of the set $\lambda(\mathcal{A}, \mathcal{B})$ defined by

$$\lambda(\mathcal{A}, \mathcal{B}) = \{\lambda \in \mathbb{H} : \mathcal{A} \star \mathcal{X} = \mathcal{B} \star \mathcal{X}\lambda\}$$

Definition (Generalized Qt-Eigenvalue Problem)

If there exists $\lambda \in \mathbb{H}$ and nonzero $\mathcal{X} \in \mathbb{H}^{n\times 1\times m}$ such that

$$\mathcal{A} \star \mathcal{X} - \mathcal{B} \star \mathcal{X}\lambda = 0$$

Then, we say that λ and $\mathcal{X}$ are the right generalized eigenvalue and the corresponding right generalized eigentensor of the pair $(\mathcal{A}, \mathcal{B})$.

Theorem

If $\lambda \in \lambda(\mathcal{A}, \mathcal{B})$, then $[\lambda] \subset \lambda(\mathcal{A}, \mathcal{B})$, where $[\lambda]$ is the equivalence class of λ.

Regular Tensor Pair Under Qt-Product

Definition

If $\mathcal{A}, \mathcal{B} \in \mathbb{H}^{n\times n\times m}$ are quaternion Qt-hermitian tensors and $\mathcal{B}$ is a Qt-positive definite tensor, then we say that $\mathcal{A}$ and $\mathcal{B}$ form a regular tensor quaternion pair.

Theorem

The regular pair $\langle \mathcal{A}, \mathcal{B} \rangle$ has mn generalized eigenvalues and all of them are **real**.

Quaternion Multilinear Discriminant Analysis (Qt-MLDA)

Data: N quaternion tensors $\mathcal{X}_i \in \mathbb{H}^{n_1 \times n_2 \times n_3}$, partitioned into C classes.

Qt-scatter operators:

$$\mathcal{S}_W = \sum_{c=1}^{C} \sum_{i \in C_c} (\mathcal{X}_i - \mathcal{M}_c) \star (\mathcal{X}_i - \mathcal{M}_c)^H,$$

$$\mathcal{S}_B = \sum_{c=1}^{C} n_c \, (\mathcal{M}_c - \mathcal{M}) \star (\mathcal{M}_c - \mathcal{M})^H.$$

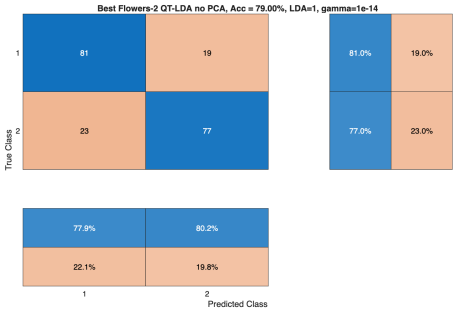
Qt-Fisher Criterion

$$\max_{\mathcal{W}} \frac{\mathcal{W}^H \star \mathcal{S}_B \star \mathcal{W}}{\mathcal{W}^H \star \mathcal{S}_W \star \mathcal{W}} \implies \boxed{\mathcal{S}_B \star \mathcal{W} \; = \; \mathcal{S}_W \star \mathcal{W} \Lambda}$$

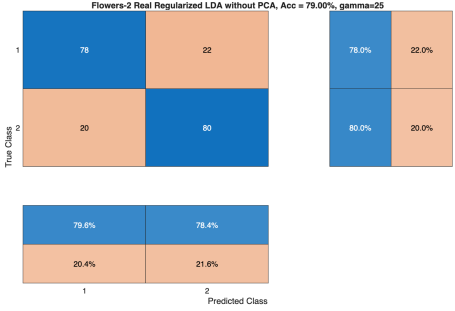
$\mathcal{S}_W$, $\mathcal{S}_B$ are Qt-Hermitian ($\mathcal{S}_W$ may need to be regularized)

Experiments: Flowers-2 (Sunflower vs. Rose), No PCA

Qt-MLDA (proposed)



Real regularized LDA



Qt-MLDA

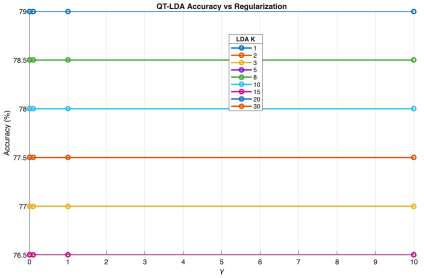
79.0% at $\gamma{=}10^{-14}$
— *no* tuning

Real LDA

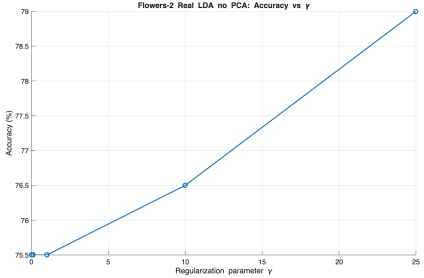
79.0% only at $\gamma{=}25$
— heavy shrinkage

no augmentation, no PCA · Qt-MLDA matches tuned real LDA at $\gamma{=}10^{-14}$.

Why It Works: Regularization Insensitivity



Qt-MLDA: accuracy **flat** in γ for every K



Real LDA: climbs 75.5% $\rightarrow$ 79% as $\gamma \rightarrow 25$

Interpretation

The Qt-product spreads class structure across the m **Fourier slices**, keeping each $\mathcal{S}_W$ slice **full-rank** and well-conditioned with a **real** spectrum. The low-rank (singular-scatter) problem is avoided — so **no or only weak regularization** is needed, whereas real LDA without PCA stays rank-deficient and heavily rely on on γ (regularization).

Qt-MLDA vs. Real LDA: Summary

Method	Accuracy	PCA?	Best γ	γ-sensitivity
Real regularized LDA	79.0%	No	25	High
Qt-MLDA (proposed)	**79.0%**	**No**	10^{-14}	**Negligible**

What the Flowers-2 experiment shows:

1. **Avoids the low-rank problem** — each Fourier slice keeps $\mathcal{S}_W$ full-rank and well-conditioned, so no PCA pre-reduction is required
2. **No (or only weak) regularization** — accuracy is flat in γ; matches tuned Qt-MLDA at γ=10^{-14}, while real LDA needs γ=25
3. **Cross-channel correlation preserved** — quaternion encoding

Summary and Future Directions

This work:

- ✓ Generalized Qt-eigenvalue theory for third-order quaternion tensors
- ✓ Real-spectrum guarantee for regular Qt-tensor pairs
- ✓ Qt-MLDA formulation with quaternion scatter operators
- ✓ Slice-wise DFT algorithm
- ✓ Flowers-2 experiment: Qt-MLDA matches regularized LDA with high regularization.

Future directions:

- Qt-MLDA full numerical comparison on color image datasets
- Qt-MLDA classification score for **outlier / anomaly detection**
- **Nonlinear extension:** Kernel Qt-MLDA and Qt-MICA
- Multimodel information fusion and classfication
- Faster computation method for this problem

Thank you!

Questions and comments are welcome.

jtian@unr.edu

Department of Mathematics and Statistics
University of Nevada, Reno